A Moment Map for the $\mathbb{T}$-action and Commuting Toeplitz Operators acting on Bergman Spaces of the Bounded Symmetric Domain of Type IV

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Seminar on Toeplitz Operators, CINVESTAV
December, 15 2021

Review and Objective

- Bounded symmetric domain of type IV:

$$\begin{aligned}\mathcal{D}_{IV} &= {}^{\mathrm{SO}_0(n,2)}\!\big/_{\mathrm{SO}(n)\times\mathrm{SO}(2)} \\ &= \left\{ z = (z_1, \cdots, z_n)^t \in \mathbb{C}^n \,\middle|\, \|z\|^2 < 1 \text{ and } 1 + |z^t z|^2 - 2\|z\|^2 > 0 \right\}\end{aligned}$$

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  has genus n.

- The group $\mathrm{SO}(n)\times\mathrm{SO}(2)$ acts on $\mathcal{D}_{IV}$ by

$$(A,\theta)z = e^{i\theta}Az$$

  where $A \in \mathrm{SO}(n)$, $\begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix} \in \mathrm{SO}(2)$ with $\theta \in \mathbb{R}$ and $z \in \mathcal{D}_{IV}$.

- This action yields the continuous unitary representation of $\mathrm{SO}(n) \times \mathrm{SO}(2)$ on $\mathcal{H}^2_\lambda(\mathcal{D}_{IV})$ given by

$$(\pi_\lambda(A,\theta)f)(z) = f((A,\theta)^{-1}z),$$

$\forall (A,\theta) \in \mathrm{SO}(n) \times \mathrm{SO}(2), f \in \mathcal{H}^2_\lambda(\mathcal{D}_{IV}), z \in \mathcal{D}.$

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$\forall (A,\theta) \in \mathrm{SO}(n) \times \mathrm{SO}(2), f \in \mathcal{H}^2_\lambda(\mathcal{D}_{IV}), z \in \mathcal{D}.$

- For $\lambda > n-1$, the C^*-algebras $\mathscr{T}_\lambda(\mathcal{A}^{\mathrm{SO}(n)\times\mathrm{SO}(2)})$ generated by Toeplitz operators with $\mathrm{SO}(n) \times \mathrm{SO}(2)$-invariant symbols are commutative.

Proposition 1

Let $\mathcal{D} = {}^{G}\!/_{K}$ be a bounded symmetric domain with genus p and $H \leq K$ be closed. For every $\lambda > p - 1$, the algebras $\mathscr{T}_\lambda(\mathcal{A}^H)$ are commutative if and only if

$$\mathcal{H}^2_\lambda(\mathcal{D}) = \bigoplus_{j \in J} V_j$$

decompose into inequivalent irrducible H-submodules. This decomposition is called the isotypic decomposition.

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When $H = K$, the corresponding Toeplitz operators T_a, $a \in \mathcal{A}^K$ are simultaneously diagonalizable. $T_a : \bigoplus_{j\in J} V_j \to \bigoplus_{j \in J} V_j$

$$T_a = \begin{array}{c@{}c} \begin{array}{ccc} V_1 & V_2 & \cdots \end{array} & \\ \begin{pmatrix} c_1(T_a)\mathrm{Id}_{V_1} & 0 & \\ 0 & c_2(T_a)\mathrm{Id}_{V_2} & \\ & & \ddots \end{pmatrix} & \begin{array}{c} V_1 \\ V_2 \\ \vdots \end{array} \end{array}$$

$c_j(T_a)$: spectrum of T_a.

Let $H \leq \mathrm{SO}(n) \times \mathrm{SO}(2)$.

- Moment Map Symbols or μ^H-symbol: $a = f \circ \mu^H$ where μ^H is a moment map for the H-action on $\mathcal{D}_{IV}$ and f is any function such that $f \circ \mu^H \in L^\infty(\mathcal{D}_{IV})$.
- The set of μ^H-symbol: $\mathcal{A}^{\mu^H}$.
- Corresponding C^*-algebras: $\mathscr{T}_\lambda(\mathcal{A}^{\mu^H})$ for every $\lambda > n - 1$.

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Objective: Find a subgroup $H \leq \mathrm{SO}(n) \times \mathrm{SO}(2)$ such that Toeplitz operators with μ^H-symbols generate commutative C^*-algebras for every $\lambda > n-1$.

Contents

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Moment Map for the $\mathbb{T}$-action on $\mathcal{D}_{IV}$

The Kähler form of $\mathcal{D}_{IV}$

- The bounded symmetric domain of type IV has a realization

$$\mathcal{D}_{IV} = \left\{ z = (z_1, \cdots, z_n)^t \in \mathbb{C}^n \,\middle|\, \|z\|^2 < 1 \text{ and } 1 + |z^t z|^2 - 2\|z\|^2 > 0 \right\}$$

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- Bergman Kernel:

$$B(z, z) = (1 - 2\|z\|^2 + |z^t z|^2)^{-n} := \Delta(z)^{-n}$$

where $\Delta(z) = 1 - 2\|z\|^2 + |z^t z|^2$.

Identifying $\mathbb{C}^n \cong \mathbb{R}^{2n}$ by

$$(z_1, \cdots, z_n) \mapsto (x_1, \cdots, x_n, y_1, \cdots, y_n)$$

where $z_j = x_j + iy_j \in \mathbb{C}$.

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- $\mathcal{D}_{IV}$ is an almost complex manifold with an almost complex structure J.
- The kernel function B induces the Riemannian metric g that is J-invariant, i.e.

$$g(J\cdot, J\cdot) = g(\cdot, \cdot).$$

  This implies that $(\mathcal{D}_{IV}, J, g)$ is a Hermitian manifold and a Riemannian metric g is referred to as a Hermitian metric.

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This implies that $(\mathcal{D}_{IV}, J, g)$ is a Hermitian manifold and a Riemannian metric g is referred to as a Hermitian metric.
- Associated to the Hermitian metric g, there is a non-degenerate closed 2-form ω given by
$$\omega(\cdot, \cdot) = g(J\cdot, \cdot)$$
and is compatible with J, i.e.
$$\omega(J(\cdot), J(\cdot)) = \omega(\cdot, \cdot).$$

- The 2-form ω is called the **Kähler form**.
  Its complexified $\omega_z : T_z^{\mathbb{C}}\mathcal{D}_{IV} \times T_z^{\mathbb{C}}\mathcal{D}_{IV} \to \mathbb{C}$ is given by

$$\omega_z = i \sum_{j,k=1}^{n} g_{jk}(z) dz_j \wedge d\overline{z}_k$$

where

$$dz_j \wedge d\overline{z}_k = dz_j \otimes d\overline{z}_k - d\overline{z}_k \otimes dz_j.$$

for any $z \in \mathcal{D}_{IV}$.

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  for any $z \in \mathcal{D}_{IV}$.

- For simplicity, we choose $g_{jk}(z)$ such that

$$g_{jk}(z) = \frac{1}{2n} \frac{\partial^2}{\partial z_j \partial \overline{z}_k} \log B(z, z).$$

Explicitly,

$$g_{jk}(z) = \frac{\Delta(z)(\delta_{jk} - 2z_j\overline{z}_k) + 2(\overline{z}_j - z_j(\overline{z^t z}))(z_k - \overline{z}_k(z^t z))}{\Delta(z)^2}$$

and the Kähler form is then

$$\omega_z = i\sum_{j,k=1}^{n} \frac{\Delta(z)(\delta_{jk} - 2z_j\overline{z}_k) + 2(\overline{z}_j - z_j(\overline{z^t z}))(z_k - \overline{z}_k(z^t z))}{\Delta(z)^2} dz_j \wedge d\overline{z}_k$$

where $z \in \mathcal{D}_{IV}$ and $\Delta(z) = 1 - 2\|z\|^2 + |z^t z|^2$.

Moment Map for the $\mathbf{T}$-action

- $H, \mathfrak{h}$: Connected Lie group, Lie algebra.
  Assume that H acts smoothly on $\mathcal{D}_{IV}$.
- For every $X \in \mathfrak{h}$, there is a smooth vector field given by

$$X_z^\sharp = \frac{d}{dr}\Big|_{r=0} \exp(rX)z$$

  for every $z \in \mathcal{D}_{IV}$.
- If we consider $X^\sharp$ as a complex-valued function with component functions given by $X^\sharp = (f_1, \cdots, f_n)$, then the corresponding expression as a complex vector field is given by

$$X^\sharp = \sum_{j=1}^{n} \left(f_j \frac{\partial}{\partial z_j} + \overline{f}_j \frac{\partial}{\partial \overline{z}_j} \right).$$

Definition 1

Let H be a connected Lie group acting smoothly on $\mathcal{D}_{IV}$ preserving ω, i.e. $\omega(dh(\cdot), dh(\cdot)) = \omega(\cdot, \cdot)$ where $dh_z : T_z\mathcal{D}_{IV} \to T_{hz}\mathcal{D}_{IV}$ is the differential of the action for any $h \in H$ at $z \in \mathcal{D}_{IV}$. Let $\mathfrak{h}$ be the Lie algebra of H and $\mathfrak{h}^*$ be its real dual space. A moment map for the $H-$action on $\mathcal{D}_{IV}$ is a smooth function

$$\mu : \mathcal{D}_{IV} \to \mathfrak{h}^*$$

that satisfies the following properties:

i. For every $X \in \mathfrak{h}$, the smooth function

$$\mu_X : \mathcal{D}_{IV} \to \mathbb{R}$$

defined by $\mu_X(z) = \langle \mu(z), X \rangle = \mu(z)(X)$ has Hamiltonian vector field given by $X^\sharp$. In other word, we have $X^\sharp = X_{\mu_X}$ for every $X \in \mathfrak{h}$.

ii. For every $h \in H$, we have

$$\mu \circ h = \mathrm{Ad}^*(h) \circ \mu$$

where Ad is the adjoint representation of H and $\mathrm{Ad}^*(h)$ denotes the transpose transformation of $\mathrm{Ad}(h)^{-1}$.

Observation 1

For every real-valued smooth function f on $\mathcal{D}_{IV}$, the Hamiltonian vector field associated to f is the smooth vector field X_f that satisfies $df = \omega(X_f, \cdot)$. Hence, the first condition in the above definition is equivalent to the requiring that μ_X satisfies

$$d\mu_X = \omega(X^\sharp, \cdot) \tag{1}$$

If H is an abelian Lie group, then $\mathrm{Ad}(h) = \mathrm{Id}_{\mathfrak{h}}$ for every $h \in H$. Thus, the second condition is equivalent to $H-$invariance of μ, i.e.

$$\mu \circ h = \mu \text{ for all } h \in H. \tag{2}$$

Suppose that $\mathcal{D}_{IV} \subseteq \mathbb{C}^{2n+1}$. A maximal torus $\mathbf{T}$ of $\mathrm{SO}(2n+1) \times \mathrm{SO}(2)$ is given by $T \times \mathrm{SO}(2)$ where

$$T = \left\{ \begin{pmatrix} \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix} & & & \\ & \ddots & & \\ & & \begin{pmatrix} \cos\theta_n & \sin\theta_n \\ -\sin\theta_n & \cos\theta_n \end{pmatrix} & \\ & & & 1 \end{pmatrix} \right\}$$

The action of $\mathbf{T}$ on $\mathcal{D}_{IV}$ is given by

$$(A, \theta) \cdot z = e^{i\theta} A z$$

where $A \in T$ and $\theta \in \mathbb{R}$.

Lemma 1

Suppose that $\mathcal{D}_{IV} \subset \mathbb{C}^N$ where $N = 2n$ or $2n+1$ and $X_1, \cdots, X_{n+1}$ is the canonical basis of the Lie algebra $\mathfrak{t}$ of $\mathbf{T}$. Suppose that $f_j : \mathcal{D}_{IV} \to \mathbb{R}$ where $j = 1, \cdots, n+1$ are smooth such that

(i). $df_j(z) = \omega(X_j^\sharp(z), \cdot)$,

(ii). $f_j \circ h = f_j$ for all $h \in \mathbf{T}$ and $z \in \mathcal{D}_{IV}$,

where $X_j^\sharp$ are the complex vector fields corresponding to X_j. Identifying $\mathfrak{t}^* \cong \mathfrak{t}$ then

$$\mu(z) = \sum_{j=1}^{n+1} f_j(z) X_j$$

satisfies (1) and (2) above, i.e. μ is a moment map for the $\mathbf{T}$-action on $\mathcal{D}_{IV}$.

Let $\mathfrak{t}$ be the Lie algebra of $\mathbf{T}$. Computing

$$X_z^\sharp = \frac{d}{dr}\Big|_{r=0} \exp(rX)z$$

where $X \in \mathfrak{t}, z \in \mathcal{D}_{IV}$ and $r \in \mathbb{R}$, then $X_z^\sharp$ as a complex vector field is given by

$$X_z^\sharp = \sum_{j=1}^{n} \left(\theta_j z_{2j} \frac{\partial}{\partial z_{2j-1}} - \theta_j z_{2j-1} \frac{\partial}{\partial z_{2j}} + \theta_j \overline{z}_{2j} \frac{\partial}{\partial \overline{z}_{2j-1}} - \theta_j \overline{z}_{2j-1} \frac{\partial}{\partial \overline{z}_{2j}} \right) + \sum_{j=1}^{2n+1} \left(i\theta z_j \frac{\partial}{\partial z_j} - i\theta \overline{z}_j \frac{\partial}{\partial \overline{z}_j} \right).$$

Let $\{X_j\}_{j=1}^{n+1}$ be the canonical basis of $\mathfrak{t} \subseteq \mathfrak{so}(2n+1) \oplus \mathfrak{so}(2)$. Then for $j = 1, \cdots, n$,

$$X_j^\sharp(z) = z_{2j} \frac{\partial}{\partial z_{2j-1}} - z_{2j-1} \frac{\partial}{\partial z_{2j}} + \overline{z}_{2j} \frac{\partial}{\partial \overline{z}_{2j-1}} - \overline{z}_{2j-1} \frac{\partial}{\partial \overline{z}_{2j}}$$

and

$$X_{n+1}^\sharp(z) = \sum_{j=1}^{2n+1} \left(i z_j \frac{\partial}{\partial z_j} - i \overline{z}_j \frac{\partial}{\partial \overline{z}_j} \right).$$

On the other hand,

$$\omega(X_j^\sharp(z),\cdot) = \sum_{k=1}^{2n+1} (-i\overline{z}_{2j}g_{k,2j-1}(z) + i\overline{z}_{2j-1}g_{k,2j}(z))dz_k + \sum_{k=1}^{2n+1} (iz_{2j}g_{2j-1,k}(z) - iz_{2j-1}g_{2j,k}(z))d\overline{z}_k \text{ for } j=1,\cdots,n$$

$$\omega\left(X_{n+1}^\sharp(z),\cdot\right) = \sum_{k=1}^{2n+1}\left(-\sum_{j=1}^{2n+1}\overline{z}_j g_{kj}(z)dz_k - \sum_{j=1}^{2n+1} z_j g_{jk}(z)d\overline{z}_k\right).$$

Since

$$df_j = \sum_{k=1}^{2n+1} \left(\frac{\partial f_j}{\partial z_k} dz_k + \frac{\partial f_j}{\partial \overline{z}_k} d\overline{z}_k \right),$$

the condition $df_j(z) = \omega(X_j^\sharp(z), \cdot)$ is equivalent to satisfy for every $k = 1, 2, \cdots, 2n+1$ the equations:

$$\frac{\partial f_j}{\partial z_k}(z) = -i\overline{z}_{2j} g_{k,2j-1}(z) + i\overline{z}_{2j-1} g_{k,2j}(z)$$
$$\frac{\partial f_j}{\partial \overline{z}_k}(z) = iz_{2j} g_{2j-1,k}(z) - iz_{2j-1} g_{2j,k}(z)$$

for $j = 1, \cdots, n$ and

$$\frac{\partial f_{n+1}}{\partial z_k}(z) = -\sum_{j=1}^{2n+1} \overline{z}_j g_{kj}(z)$$

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Theorem 1

Identifying $\mathfrak{t}$ with $\mathbb{R}^{n+1}$, a moment map for the $\mathbf{T}$-action on

$$\mathcal{D}_{IV} = \left\{ z = (z_1, \cdots, z_n)^t \in \mathbb{C}^n \middle| \|z\|^2 < 1 \text{ and } 1 + |z^t z|^2 - 2\|z\|^2 > 0 \right\}$$

is the smooth function $\mu : \mathcal{D}_{IV} \to \mathbb{R}^{n+1}$ given by

$$\mu(z) = \frac{1}{\Delta(z)} \sum_{j=1}^{n} i(\overline{z}_{2j-1} z_{2j} - z_{2j-1}\overline{z}_{2j}) e_j + \frac{1}{\Delta(z)} (|z^t z|^2 - \|z\|^2) e_{n+1}$$

where $z \in \mathcal{D}_{IV}, \Delta(z) = 1 - 2\|z\|^2 + |z^t z|^2$ and $\{e_j\}_j$ is the canonical basis of $\mathbb{R}^{n+1}$.

Toeplitz Operators with $\mu^{\mathrm{SO}(2)}$-symbols on $\mathcal{D}_{IV}$

Toeplitz Operators with $\mu^{\mathrm{SO}(2)}$-symbols

Proposition 2

Let $\mathbf{T}$ be the maximal torus of $\mathrm{SO}(n) \times \mathrm{SO}(2)$ and $\mu^{\mathbf{T}} : \mathcal{D}_{IV} \to \mathbb{R}^{n+1}$ be the moment map of $\mathbf{T}$ for $\mathcal{D}_{IV}$ given in Theorem 1. Suppose that H is a connected subgroup of $\mathbf{T}$. Then a moment map for the H-action on $\mathcal{D}_{IV}$ is given by

$$\mu^H : \mathcal{D}_{IV} \to \mathfrak{h}$$
$$\mu^H = \iota^* \circ \mu^{\mathbf{T}}$$

where ι^* is the orthogonal projection $\mathbb{R}^{n+1} \to \mathfrak{h}$. In particular, the $\mathrm{SO}(2)$-action on $\mathcal{D}_{IV}$ has a moment map given by

$$\mu^{\mathrm{SO}(2)} : \mathcal{D}_{IV} \to \mathbb{R}$$
$$\mu^{\mathrm{SO}(2)}(z) = \frac{|z^t z|^2 - \|z\|^2}{1 - 2\|z\|^2 + |z^t z|^2}.$$

Theorem 2

Suppose that $\mathcal{D}_{IV} \subseteq \mathbb{C}^n$ where $n \geq 3$. Let $\mu^{\mathrm{SO}(2)} : \mathcal{D}_{IV} \to \mathbb{R}$ be a moment map for $\mathrm{SO}(2)$ on $\mathcal{D}_{IV}$ given in the previous proposition. Then for $\lambda > n-1$, the Toeplitz C^*-algebras $\mathscr{T}_\lambda(\mathcal{A}^{\mu^{\mathrm{SO}(2)}})$ are commutative.

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Proof.

$$\mathcal{A}^{\mu^{\mathrm{SO}(2)}} \subseteq \mathcal{A}^{\mathrm{SO}(n)\times\mathrm{SO}(2)}.$$

Then

$$\mathscr{T}_\lambda(\mathcal{A}^{\mu^{\mathrm{SO}(2)}}) \subseteq \mathscr{T}_\lambda(\mathcal{A}^{\mathrm{SO}(n)\times\mathrm{SO}(2)}).$$

$\square$

Spectral Integral Formulas

How to compute the spectral integral formulas for Toeplitz operators with $\mu^{\mathrm{SO}(2)}$-symbols?

Spectral Integral Formulas

How to compute the spectral integral formulas for Toeplitz operators with $\mu^{\mathrm{SO}(2)}$-symbols?

Since $\mathcal{A}^{\mu^{\mathrm{SO}(2)}} \subseteq \mathcal{A}^{\mathrm{SO}(n)\times\mathrm{SO}(2)}$, we can use the formula in [DQB18].

- $\mathcal{D}_{IV} \subseteq \mathbb{C}^n$ with $n \geq 3$.
- We have a decomposition into irreducible $\mathrm{SO}(n) \times \mathrm{SO}(2)$-submodules with multiplicity 1

$$\mathcal{P}(\mathbb{C}^n) = \bigoplus_{(k_1,k_2)\in\mathbb{N}^2} V_{k_1,k_2}.$$

- This induces the multiplicity-free isotypic decomposition for the representation of $\mathrm{SO}(n) \times \mathrm{SO}(2)$ on the Bergman space $\mathcal{H}^2_\lambda(\mathcal{D}_{IV})$ where $\lambda > n-1$.

$$\mathcal{H}^2_\lambda(\mathcal{D}_{IV}) = \bigoplus_{(k_1,k_2)\in\mathbb{N}^2} V_{k_1,k_2}.$$

- If $a \in \mathcal{A}^{\mathrm{SO}(n)\times\mathrm{SO}(2)}$, the Toeplitz operator $T_a : \mathcal{H}^2_\lambda(\mathcal{D}_{IV}) \to \mathcal{H}^2_\lambda(\mathcal{D}_{IV})$ when restricts to V_{k_1,k_2} for every $(k_1,k_2) \in \mathbb{N}^2$ is

$$T_a\big|_{V_{k_1,k_2}} = c_{k_1,k_2}(T_a)\mathrm{Id}_{V_{k_1,k_2}}$$

Furthermore,

$$c_{k_1,k_2}(T_a) = \frac{\langle T_a\varphi, \varphi\rangle_\lambda}{\langle \varphi, \varphi\rangle_\lambda} \tag{3}$$

for every non-zero $\varphi \in V_{k_1,k_2}$ and $\langle f, g\rangle_\lambda = \int f\overline{g}dv_\lambda$ is the inner product on $\mathcal{H}^2_\lambda(\mathcal{D}_{IV})$.

Furthermore,

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For each $(k_1, k_2) \in \mathbb{N}^2$, the irreducible $\mathrm{SO}(n) \times \mathrm{SO}(2)$-submodule V_{k_1,k_2} corresponds with the highest weight vector

$$p_{k_1,k_2}(z) = p_1(z)^{k_1} p_2(z)^{k_2} = (z_1 - iz_2)^{k_1}(z_1^2 + \cdots + z_n^2)^{k_2}$$

where $z = (z_1, \cdots, z_n)^t \in \mathbb{C}^n$.

There is a Jordan algebra associated with the domain $\mathcal{D}_{IV}$ given as follows:

- Write $\mathbb{R}^n = \mathbb{R}e_1 \oplus \mathbb{R}^{n-1}$ where $e_1 = (1, 0, \cdots, 0)^t \in \mathbb{R}^n$ and for any $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, we write $x = x_1 e_1 + x'$ where $x' = (x_2, \cdots, x_n)^t \in \mathbb{R}^{n-1}$.
- $(\mathbb{R}e_1 \oplus \mathbb{R}^{n-1}, \circ)$ is a Jordan algebra with the unit element e_1 where

$$(x_1 e_1 + x') \circ (y_1 e_1 + y') = (x_1 y_1 + x' \cdot y') e_1 + (x_1 y' + y_1 x')$$

- Cone of positive elements: $x^2 = x \circ x$

$$\begin{aligned} \Omega &= \{x^2 | x \in \mathbb{R}^n\}^\circ \\ &= \{x_1 e_1 + x' | x_1 > 0, x_1^2 - x' \cdot x' > 0\} \end{aligned}$$

- Order on $\mathbb{R}^n$: $x \succ 0$ if $x \in \Omega$

- A root of x: Any $y \in \mathbb{R}^n$ such that $y^2 = x$.

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Proposition 3

For any $x = x_1 e_1 + x' \in \Omega$, there is the unique root denoted by $\sqrt{x} \in \Omega$ and it is given by

$$\sqrt{x} = \sqrt{\frac{x_1 + \sqrt{x_1^2 - x' \cdot x'}}{2}} e_1 + \frac{x'}{\sqrt{2(x_1 + \sqrt{x_1^2 - x' \cdot x'})}}.$$

- The embedding:

$$E : \mathbb{R}^n \to \mathbb{C}^n, E(x_1 e_1 + x') = x_1 e_1 + ix'$$

- $(E(\mathbb{R}^n), \circ)$ is a Jordan algebra with the unit e_1 and the product is given by:

$$(x_1 e_1 + ix') \circ (y_1 e_1 + iy') = (x_1 y_1 + x' \cdot y')e_1 + i(x_1 y' + y_1 x')$$

- Cone of positive elements:

$$\widehat{\Omega} = E(\Omega) = \{x_1 e_1 + ix' | x_1 > 0, x_1^2 - x' \cdot x' > 0\}$$

- Write $\mathbb{C}^n = E(\mathbb{R}^n) \oplus iE(\mathbb{R}^n)$, then $\mathbb{C}^n$ becomes a complex Jordan algebra with involution * given by $(x+iy)^* = x - iy$ with $x, y \in E(\mathbb{R}^n)$. The product is given by

$$z \circ w = (z_1 w_1 - z' \cdot w')e_1 + (z_1 w' + w_1 z')$$

  where $z = z_1 e_1 + z', w = w_1 e_1 + w'$ with $z_1, w_1 \in \mathbb{C}, z', w' \in \mathbb{C}^{n-1}$.
- $(\mathbb{C}^n, \circ)$ is a complex Jordan algebra with the unit e_1 associated with the domain $\mathcal{D}_{IV}$.

For any $(k_1, k_2) \in \mathbb{N}^2$, We have the non-zero polynomial that belongs to V_{k_1,k_2} given by

$$\phi_{k_1,k_2}(z) = \int_L p_{k_1,k_2}(gz)dg$$

where dg is the normalized Haar measure on L and L is the identity component of the isotropy subgroup of automorphism of the cone $\widehat{\Omega}$ that fixes the unit e_1.

From Theorem 4.11 in [DQB18], the coefficients $c_{k_1,k_2}(T_a)$ in (3) for a $\mathrm{SO}(n) \times \mathrm{SO}(2)$-invariant symbol a is given by

$$\begin{aligned} c_{k_1,k_2}(T_a) &= \frac{\langle T_a \phi_{k_1,k_2}, \phi_{k_1,k_2} \rangle_\lambda}{\langle \phi_{k_1,k_2}, \phi_{k_1,k_2} \rangle_\lambda} \\ &= \frac{\int_{\widehat{\Omega} \cap (e_1 - \widehat{\Omega})} a(\sqrt{x}) p_1(x)^{k_1} p_2(x)^{k_2} p_2(e_1 - x)^{\lambda - n} dx}{\int_{\widehat{\Omega} \cap (e_1 - \widehat{\Omega})} p_1(x)^{k_1} p_2(x)^{k_2} p_2(e_1 - x)^{\lambda - n} dx} \\ &= \frac{\int_{\Omega \cap (e_1 - \Omega)} a(E(\sqrt{x})) p_1(E(x))^{k_1} p_2(E(x))^{k_2} p_2(e_1 - E(x))^{\lambda - n} dx}{\int_{\Omega \cap (e_1 - \Omega)} p_1(E(x))^{k_1} p_2(E(x))^{k_2} p_2(e_1 - E(x))^{\lambda - n} dx} \end{aligned}$$

where

$$
\begin{aligned}
x &= (x_1, x_2, \cdots, x_n)^t = x_1 e_1 + x' \in \mathbb{R}^n, \\
dx &= dx_1 dx_2 \cdots dx_n \text{ is the Lebesgue measure,} \\
p_1(E(x)) &= p_1(x_1 e_1 + i x') = x_1 + x_2, \\
p_2(E(x)) &= p_2(x_1 e_1 + i x') = x_1^2 - x_2^2 + \cdots - x_n^2 = x_1^2 - x' \cdot x', \\
p_2(e_1 - E(x)) &= (1 - x_1)^2 - x' \cdot x', \\
E(\sqrt{x}) &= \sqrt{\frac{x_1 + \sqrt{x_1^2 - x' \cdot x'}}{2}} e_1 + \frac{i x'}{\sqrt{2(x_1 + \sqrt{x_1^2 - x' \cdot x'})}}, \\
\Omega \cap (e_1 - \Omega) &= 0 \prec x \prec e_1 \\
&= \left\{ x_1 e_1 + x' \in \mathbb{R} e_1 \oplus \mathbb{R}^{n-1} \,\middle|\, \begin{array}{c} 0 < x_1 < 1 \\ \sqrt{x' \cdot x'} < x_1 < 1 - \sqrt{x' \cdot x'} \end{array} \right\}
\end{aligned}
$$

Theorem 3

Suppose that $\mathcal{D}_{IV} \subseteq \mathbb{C}^n$ $(n \geq 3)$ is given by

$$\mathcal{D}_{IV} = \left\{ z = (z_1, \cdots, z_n)^t \in \mathbb{C}^n \,\middle|\, \|z\|^2 < 1 \text{ and } 1 + |z^t z|^2 - 2\|z\|^2 > 0 \right\}.$$

Let $f \circ \mu^{\mathrm{SO}(2)} \in \mathcal{A}^{\mu^{\mathrm{SO}(2)}}$ be a moment map symbol. For every $\lambda > n-1$ and $(k_1, k_2) \in \mathbb{N}^2$, the spectral integral formula for the Toeplitz operator $T_{f \circ \mu^{\mathrm{SO}(2)}}$ is given by

$$c_{k_1,k_2}(T_{f\circ\mu^{\mathrm{SO}(2)}}) = \frac{\displaystyle\int_{0 \prec x \prec e_1} f\left(\frac{x'\cdot x'}{\|x\|^2 - 1}\right)(x_1+x_2)^{k_1}(x_1^2 - x'\cdot x')^{k_2}((1-x_1)^2 - x'\cdot x')^{\lambda-n}dx}{\displaystyle\int_{0 \prec x \prec e_1} (x_1+x_2)^{k_1}(x_1^2 - x'\cdot x')^{k_2}((1-x_1)^2 - x'\cdot x')^{\lambda-n}dx}$$

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Thank You!