Toeplitz Operators With Invariant Symbols Acting on Bergman Spaces of the Bounded Symmetric Domain of Type IV

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Areas of Research and Objective

- Bounded symmetric domain: $\mathcal{D} = {}^{G}\!/_{K} \subset \mathbb{C}^n$ has genus p.
  Example: unit disk, complex unit ball.
- (Weightless) Bergman Space: $\mathcal{H}^2(\mathcal{D}) = L^2(\mathcal{D}, dv) \cap \mathcal{O}(\mathcal{D})$.
  Bergman Kernel: $B : \mathcal{D} \times \mathcal{D} \to \mathbb{C}$.
- Weighted Bergman Space: For $\lambda > p - 1$,

$$\mathcal{H}^2_\lambda(\mathcal{D}) = L^2(\mathcal{D}, dv_\lambda) \cap \mathcal{O}(\mathcal{D})$$

  Bergman Kernel: $B_\lambda(z, w) = B(z, w)^{\frac{\lambda}{p}}$
- Bergman projection:

$$P_\lambda f(z) = \int_{\mathcal{D}} f(w) B_\lambda(z, w) dv_\lambda(w).$$

- Toeplitz Operator: For any $a \in L^\infty(\mathcal{D})$ called symbol, the Toeplitz operator $T_a^{(\lambda)} : \mathcal{H}_\lambda^2(\mathcal{D}) \to \mathcal{H}_\lambda^2(\mathcal{D})$ is defined by

$$T_a^{(\lambda)}(f) = P_\lambda(af), f \in \mathcal{H}_\lambda^2(\mathcal{D})$$

In particular, for every $f \in \mathcal{H}_\lambda^2(\mathcal{D})$ and $z \in \mathcal{D}$,

$$T_a^{(\lambda)}(f)(z) = \int_{\mathcal{D}} a(w)f(w)B_\lambda(z,w)dv_\lambda(w)$$

T_a is bounded and thus is in $\mathcal{B}(\mathcal{H}_\lambda^2(\mathcal{D}))$: the set of bounded operators on $\mathcal{H}_\lambda^2(\mathcal{D})$ which is a C^*-algebra.

Interesting C^*-algebras: C^*-algebras generated by Toeplitz operators.

Problem: Find the set of symbol $\mathcal{A}$ such that the corresponding Toeplitz operators $\{T_a : a \in \mathcal{A}\}$ generate commutative C^*-algebras.

Two interesting symbols:

- Invariant Symbols: Unit disk, unit ball,$\cdots$, Bounded Symmetric Domains.
- Moment Map Symbols: Unit ball.

The biholomorphism group G on $\mathcal{D}$ acts on $L^\infty(\mathcal{D})$ by $(g\cdot f)(z) = f(g^{-1}z)$ for any $g \in G, f \in L^\infty(\mathcal{D})$ and $z \in \mathcal{D}$.
Let $H \leq G$. A symbol a is said to be H-invariant if

$$a \circ h = a, \forall h \in H.$$

- $\mathcal{A}^H$: The set of H-invariant symbols.
- $\mathscr{T}_\lambda(\mathcal{A}^H)$: The C^*-algebra generated by the family of Toeplitz operators with H-invariant symbols

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Known Result: If $H = K$ then $\mathscr{T}_\lambda(\mathcal{A}^K)$ are commutative.

- Bounded Symmetric Domain of Type IV:

$$\mathcal{D}_{IV} = {}^{\mathrm{SO}_o(n,2)}\!/_{\mathrm{SO}(n)\times\mathrm{SO}(2)} \subseteq \mathbb{C}^n$$

has genus n.

For every $\lambda > n-1$

$$\mathscr{T}_\lambda(\mathcal{A}^{\mathrm{SO}(n)\times\mathrm{SO}(2)}) \text{ are commutative.}$$

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Observation: If $H \leq K$ then $\mathcal{A}^K \subseteq \mathcal{A}^H$. Therefore, for any $\lambda > n-1$,

$$\mathscr{T}_\lambda(\mathcal{A}^K) \subseteq \mathscr{T}_\lambda(\mathcal{A}^H).$$

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Objective: Study Toeplitz operators with $\mathbf{T}$-invariant symbols and with $\mathrm{SO}(n-1)\times\mathrm{SO}(2)$-invariant symbols where $\mathbf{T}$ denote a maximal torus of $\mathrm{SO}(n)\times\mathrm{SO}(2)$.

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   - Application of Representation Theory
   - The C^*-algebras $\mathscr{T}_\lambda(\mathcal{A}^{\mathbf{T}})$
2. Toeplitz Operators with $\mathrm{SO}(n-1)\times\mathrm{SO}(2)$-invariant Symbols
   - The Highest Weight Theorem and Branching Rule
   - The C^*-algebras $\mathscr{T}_\lambda(\mathcal{A}^{\mathrm{SO}(n-1)\times\mathrm{SO}(2)})$

Toeplitz Operators with $\mathbf{T}$-invariant Symbols

Application of Representation Theory

Let $\mathcal{D} = {}^{G}\!/_{K}$ have genus p. Suppose that $\mathcal{D}$ is circled and K is the isotopy group fixing 0. The maximal compact subgroup K admits a continuous unitary representation $\pi_\lambda : K \to \mathrm{U}(\mathcal{H}^2_\lambda(\mathcal{D}))$ on $\mathcal{H}^2_\lambda(\mathcal{D})$ given by

$$(\pi_\lambda(k)f)(z) = f(k^{-1}z), \forall k \in K, f \in \mathcal{H}^2_\lambda(\mathcal{D}), z \in \mathcal{D}.$$

Let H be a closed subgroup of K.

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Proposition 1

For every $\lambda > p-1$, the algebras $\mathscr{T}_\lambda(\mathcal{A}^H)$ are commutative if and only if the unitary representations π_λ are multiplicity-free if and only if

$$\mathcal{H}^2_\lambda(\mathcal{D}) = \bigoplus_{j\in J} V_j$$

decomposes into inequivalent irreducible H-submodules V_j.

Proposition 2

Suppose that a bounded symmetric domain $\mathcal{D} = {}^{G}/_{K}$ is circled and K is the isotropy subgroup fixing 0. Then the set of polynomial in $\mathcal{H}^2_\lambda(\mathcal{D})$ is dense and π_λ-invariant in $\mathcal{H}^2_\lambda(\mathcal{D})$. In addition, the isotypic decomposition of $\mathcal{H}^2_\lambda(\mathcal{D})$ and of $\mathcal{P}(\mathbb{C}^n)$ coincide as K-modules.

Proposition 2

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We use the domain $\mathcal{D}_{IV}$ with the realization given by

$$\begin{aligned}\mathcal{D}_{IV} &= \left\{ z = (z_1, \cdots, z_n)^t \in \mathbb{C}^n \,\middle|\, \|z\|^2 < 1 \text{ and } 1 + |z^t z|^2 - 2\|z\|^2 > 0 \right\} \\ &= {}^{\mathrm{SO}_0(n,2)}\!\Big/\!_{\mathrm{SO}(n) \times \mathrm{SO}(2)}\end{aligned}$$

The C^*-algebras $\mathscr{T}_\lambda(\mathcal{A}^{\mathbf{T}})$

The action of $\mathrm{SO}(N)\times\mathrm{SO}(2)$ on the N-dimensional domain $\mathcal{D}_{IV}$ is given by

$$(A,\theta)z = e^{i\theta}Az$$

where $A\in\mathrm{SO}(N)$, $\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\in\mathrm{SO}(2)$ with $\theta\in\mathbb{R}$ and $z\in\mathcal{D}_{IV}$.

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A maximal torus of $\mathrm{SO}(N)\times\mathrm{SO}(2)$ is $\mathbf{T}=T\times\mathrm{SO}(2)$ where T is a maximal torus of $\mathrm{SO}(N)$ such that any element in T is in the following form:

For $N = 2n + 1$,

$$\begin{pmatrix} \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix} & & & \\ & \ddots & & \\ & & \begin{pmatrix} \cos\theta_n & \sin\theta_n \\ -\sin\theta_n & \cos\theta_n \end{pmatrix} & \\ & & & 1 \end{pmatrix}$$

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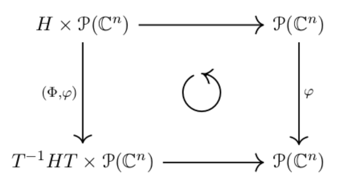
Proposition 3

Let $\mathcal{D} \subset \mathbb{C}^n$ be a bounded symmetric domain and H be a compact group acting linearly on $\mathbb{C}^n$ preserving $\mathcal{D}$. Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a linear isomorphism and $\Phi : H \to T^{-1}HT, h \mapsto T^{-1}hT$. Define:

$$\varphi : \mathcal{P}(\mathbb{C}^n) \to \mathcal{P}(\mathbb{C}^n)$$
$$p(z) \mapsto p(T^{-1}z)$$

where $p(z)$ is a complex n-variables polynomial. Then φ is an isomorphism and it is Φ-equivariant; that is, $\varphi(hp(z)) = \Phi(h)\varphi(p(z))$ for all $h \in H, p(z) \in \mathcal{P}(\mathbb{C}^n)$, i.e. the following diagram commutes.



In particular, $V \subseteq \mathcal{P}(\mathbb{C}^n)$ is an H-(irreducible) module if and only if $\varphi(V)$ is an $T^{-1}HT$-(irreducible) module.

For $N = 2n$, the Lie algebra of $\mathrm{SO}(2n)$ and its complexification are given as below.

$$\begin{aligned}\mathfrak{so}(2n) &= \left\{X \in \mathfrak{gl}_{2n}(\mathbb{R}) | X^t = -X\right\}\\ &= \left\{X = \begin{pmatrix} A & B \\ -B^t & C\end{pmatrix} \middle| A, C \in \mathfrak{so}(n), B \in \mathfrak{gl}_k(\mathbb{R})\right\},\\ \mathfrak{so}(\mathbb{C}^{2n}) &= \left\{X \in \mathfrak{gl}_{2n}(\mathbb{C}) | X^t = -X\right\}.\end{aligned}$$

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For $N = 2n+1$, the Lie algebra of $\mathrm{SO}(2n+1)$ and its complexification are given as below.

$$\begin{aligned}\mathfrak{so}(2n+1) &= \left\{X \in \mathfrak{gl}_{2n+1}(\mathbb{R}) | X^t = -X\right\} \\ &= \left\{X = \begin{pmatrix} A & B & \eta \\ -B^t & C & \gamma \\ -\eta^t & -\gamma^t & 0 \end{pmatrix} \middle| \begin{array}{c} A, C \in \mathfrak{so}(n), B \in \mathfrak{gl}_n(\mathbb{R}), \\ \eta, \gamma \in \mathbb{M}_{k\times 1}(\mathbb{R}) \end{array} \right\}, \\ \mathfrak{so}(\mathbb{C}^{2n+1}) &= \left\{X \in \mathfrak{gl}_{2n+1}(\mathbb{C}) | X^t = -X\right\}.\end{aligned}$$

We use the matrices of change of coordinates as below:

$$P_n = \frac{1}{\sqrt{2}}\begin{pmatrix} I_n & I_n \\ -iI_n & iI_n \end{pmatrix} \qquad \text{for } N = 2n$$

$$Q_n = \begin{pmatrix} P_n & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{for } N = 2n+1.$$

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We denote $S_n = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$ and $S_{n,1} = \begin{pmatrix} S_n & 0 \\ 0 & 1 \end{pmatrix}$

For $N = 2n$,

$$\mathfrak{so}(2n, S_n) = \left\{ Y = \begin{pmatrix} E & F \\ \overline{F} & \overline{E} \end{pmatrix} \in \mathfrak{gl}_{2n}(\mathbb{C}) \middle| E \in \mathfrak{u}(n), F \in \mathfrak{so}(\mathbb{C}^n) \right\}$$

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For $N = 2n + 1$,

$$\mathfrak{so}(\mathbb{C}^{2n+1}, S_{n,1}) = \left\{ Y \in \mathfrak{gl}_{2n+1}(\mathbb{C}) | Y^t S_{n,1} = -S_{n,1} Y, \mathrm{Tr}(Y) = 0 \right\},$$

$$\mathfrak{so}(2n+1, S_{n,1}) = \left\{ Y = \begin{pmatrix} E & F & \xi \\ \overline{F} & \overline{E} & \nu \\ -\nu^t & -\xi^t & 0 \end{pmatrix} \middle| \begin{array}{c} E \in \mathfrak{u}(n), F \in \mathfrak{so}(\mathbb{C}^n), \\ \xi, \nu \in \mathbb{M}_{n\times 1}(\mathbb{C}) \end{array} \right\}.$$

The maximal abelian subalgebra of $\mathfrak{so}(2n, S_n)$:

$$\mathfrak{t}_{0,S_n} = \{d(\vartheta) = \mathrm{diag}(i\vartheta_1, \cdots, i\vartheta_n, -i\vartheta_1, \cdots, -i\vartheta_n) | \vartheta_1, \cdots, \vartheta_n \in \mathbb{R}\}.$$

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$$T_{S_n} = \{\mathrm{diag}(e^{i\vartheta_1}, e^{i\vartheta_2}, \cdots, e^{i\vartheta_n}, e^{-i\vartheta_1}, e^{-i\vartheta_2}, \cdots, e^{-i\vartheta_n}) | \vartheta_1, \cdots, \vartheta_n \in \mathbb{R}\}$$

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Theorem 1

Suppose that $N \geq 3$ and $\mathcal{D}_{IV} \subseteq \mathbb{C}^N$ where $N = 2n$ or $2n+1$. With respect to the w-coordinates, i.e. with respect to the matrices of change of coordinates P_n and Q_n above, the decomposition of $\mathcal{P}(\mathbb{C}^N)$ into $\mathbf{T}$-submodules where $\mathbf{T}$ is a maximal torus of $\mathrm{SO}(N)\times\mathrm{SO}(2)$ is given by

$$\mathcal{P}(\mathbb{C}^N) = \bigoplus_{\alpha\in\mathbb{N}^N} \mathbb{C}w^\alpha$$

and it decomposes with higher multiplicity. In particular, the Toeplitz C^*-algebras $\mathscr{T}_\lambda(\mathcal{A}^{\mathbf{T}})$ where $\lambda > N-1$ are not commutative.

Proof: Computing the character of the representations.

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The action of $\mathbf{T}$ on the monomial w^α where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{2n}) \in \mathbb{N}^{2n}$:

$$\begin{aligned}\pi_\lambda((t,s))w^\alpha &= \left((t,s)^{-1}\cdot w\right)^\alpha \\ &= s^{-|\alpha|}\prod_{j=1}^{n} t_j^{\alpha_{n+j}-\alpha_j} w^\alpha.\end{aligned}$$

The character: $\chi_{\alpha,n}(t,s) = s^{-|\alpha|}\prod_{j=1}^{n} t_j^{\alpha_{n+j}-\alpha_j}$.

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Toeplitz Operators with $\mathrm{SO}(n-1)\times\mathrm{SO}(2)$-invariant Symbols

The Highest Weight Theorem and Branching Rule

Definition 1

Let $\mathfrak{h}$ be a Cartan subalgebra of a semisimple Lie algebra $\mathfrak{g}$ and α be an element of the dual $\mathfrak{h}^*$. Let φ be a representation of $\mathfrak{g}$ on a complex vector space V. We define

$$V_\alpha := \{v \in V | \varphi(x)v = \alpha(x)v, \forall x \in \mathfrak{h}\}.$$

When $V_\alpha \neq 0$, we call α a *weight* of the representation φ of $\mathfrak{g}$ with respects to the Cartan subalgebra $\mathfrak{h}$ and call V_α a *weight space*. Members of V_α are called *weight vectors*. For the adjoint representation, we have

$$\mathfrak{g}_\alpha = \{v \in V | [x, v] = \alpha(x)v, \forall x \in \mathfrak{h}\}.$$

A weight, weight space and weight vector for the adjoint representation of $\mathfrak{g}$ are respectively called a *root, root space* and *root vector*.

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The Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$: Any element $B \in \mathfrak{t}$ is a block diagonal matrix such that

$$B = \begin{pmatrix} B_1 & & & \\ & \ddots & & \\ & & B_n & \\ & & & 0 \end{pmatrix}$$

where

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Collection of roots: $\Phi = \{\pm(e_j \pm e_k), \pm e_l | 1 \leq j < k \leq n, 1 \leq l \leq n\}$.
For $\mathfrak{g} = \mathfrak{so}(\mathbb{C}^{2n})$: $\Phi = \{\pm(e_j \pm e_k) | 1 \leq j < k \leq n\}$.

Theorem of the highest weight (ideas and constructions):
$(\mathfrak{g}, \mathfrak{h})$: Lie algebra-Cartan subalgebra pair.

- Define an order on $\mathfrak{h}^* \to$ positive roots $\Phi^+ \to$ simple roots Δ.
- Define bilinear form on $\mathfrak{h}^* \times \mathfrak{h}^* \to$ algebraically integral, dominant.

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$$\left\{ \begin{matrix} (\mathfrak{g}, \mathfrak{h}) \\ \varphi : \text{irreducible representation of } \mathfrak{g} \end{matrix} \right\}$$

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$$\left\{\begin{matrix} (\mathfrak{g},\mathfrak{h}) \\ \varphi : \text{irreducible representation of } \mathfrak{g} \end{matrix}\right\} \quad \rightleftarrows \quad \left\{\begin{matrix} \text{The highest weight } \lambda \in \mathfrak{h}^* \\ \text{dominant algebraically integral} \end{matrix}\right\}$$

Theorem of the highest weight (ideas and constructions):
$(\mathfrak{g}, \mathfrak{h})$: Lie algebra-Cartan subalgebra pair.

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Proposition 4

For $\mathrm{SO}(2n+1)$, the irreducible representations correspond to the highest weights of the form $\sum_{j=1}^{n} a_j e_j$ such that

$$a_1 \geq \cdots \geq a_n \geq 0.$$

For $\mathrm{SO}(2n)$, the irreducible representations correspond to the highest weights of the form $\sum_{j=1}^{n} a_j e_j$ such that

$$a_1 \geq \cdots \geq a_{n-1} \geq |a_n|.$$

Note:
For $\mathfrak{so}(\mathbb{C}^{2n+1}) : \Delta = \{e_1 - e_2, e_2 - e_3, \cdots, e_{n-1} - e_n, e_n\}$.
For $\mathfrak{so}(\mathbb{C}^{2n}) : \Delta = \{e_1 - e_2, e_2 - e_3, \cdots, e_{n-1} - e_n, e_{n-1} + e_n\}$.

Example: $G = \mathrm{SO}(N), \mathfrak{g}_0 = \mathfrak{so}(N), \mathfrak{g} = \mathfrak{so}(\mathbb{C}^N)$.

$$V := \mathcal{P}(\mathbb{C}^N) = \bigoplus_{m=0}^{\infty} \mathcal{P}^m(\mathbb{C}^N)$$

where $\mathcal{P}^m(\mathbb{C}^N)$ is the subspace of homogeneous polynomials of degree m. The highest weight of $\mathcal{P}^m(\mathbb{C}^N)$ (related to the usual positive system) is me_1.

Branching Rules: Rules for decomposing the restriction of an irreducible representation into irreducible representations of the subgroup.
V is irreducible G-module, $H \leq G$.

$$V \xrightarrow{\text{Branching Rule}} V = \bigoplus_j V_j, \text{ where } V_j \text{ is irre. } H\text{-submodule}$$

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Embedding:

$$\mathrm{SO}(n-1) \hookrightarrow \mathrm{SO}(n)$$
$$M \mapsto \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}$$

Proposition 5 (Murnaghan)

i. For $\mathrm{SO}(2n+1)$, the irreducible representation with highest weight $a_1e_1+\cdots+a_ne_n$ decomposes with multiplicity 1 under $\mathrm{SO}(2n)$, and the representations of $\mathrm{SO}(2n)$ that appear are exactly those with highest weight $c_1e_1+\cdots+c_ne_n$ such that

$$a_1 \geq c_1 \geq a_2 \geq c_2 \geq \cdots \geq a_{n-1} \geq c_{n-1} \geq a_n \geq |c_n|.$$

ii. For $\mathrm{SO}(2n)$, the irreducible representation with highest weight $a_1e_1+\cdots+a_ne_n$ decomposes with multiplicity 1 under $\mathrm{SO}(2n-1)$, and the representations of $\mathrm{SO}(2n-1)$ that appear are exactly those with highest weight $c_1e_1+\cdots+c_{n-1}e_{n-1}$ such that

$$a_1 \geq c_1 \geq a_2 \geq c_2 \geq \cdots \geq a_{n-1} \geq c_{n-1} \geq |a_n|.$$

The C^*-algebras $\mathscr{T}_\lambda(\mathcal{A}^{\mathrm{SO}(n-1)\times\mathrm{SO}(2)})$

Suppose that $\mathcal{D}_{IV} \subseteq \mathbb{C}^n$ and $\lambda > n-1$. Then the C^*-algebras $\mathscr{T}_\lambda(\mathcal{A}^{\mathrm{SO}(n)\times\mathrm{SO}(2)})$ are commutative.

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Proposition 6 (Johnson)

The space $\mathcal{P}(\mathbb{C}^n)$ can be decomposed into irreducible $\mathrm{SO}(n) \times \mathrm{SO}(2)$-submodules with multiplicities 1. More precisely,

$$\mathcal{P}(\mathbb{C}^n) = \bigoplus_{(k_1,k_2)\in\mathbb{N}^2} V_{k_1,k_2}$$

where V_{k_1,k_2} is a subspace of $\mathcal{P}^m(\mathbb{C}^n)$ of homogeneous polynomials of degree $m = k_1 + 2k_2$, is an irreducible $\mathrm{SO}(n) \times \mathrm{SO}(2)$-submodule that occurs with multiplicity 1.

Observation: For any $(k_1, k_2) \in \mathbb{N}^2$,

- The representation of $\mathrm{SO}(2)$ on V_{k_1,k_2} corresponds to the character $t \mapsto t^{-(k_1+2k_2)}$ where $t \in \mathrm{SO}(2) \cong \mathbb{T}$.
- The representation of $\mathrm{SO}(n)$ on V_{k_1,k_2} corresponds to the highest weight $k_1 e_1$ where e_1 defined as before. Thus,

$$V_{k_1,k_2} \cong V_{k_1 e_1} \text{ as } \mathrm{SO}(n)\text{-modules}$$

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$$V_{k_1,k_2} \cong V_{k_1 e_1} \text{ as } \mathrm{SO}(n)\text{-modules}$$

- From the Branching rule $\mathrm{SO}(n) \downarrow \mathrm{SO}(n-1)$, we have

$$V_{k_1 e_1} = \bigoplus_{c_1=0}^{k_1} V_{c_1 e_1}$$

where $V_{c_1 e_1}$ is the irreducible $\mathrm{SO}(n-1)$-submodule corresponds to the highest weight $c_1 e_1$.

Theorem 2

Suppose that $\mathcal{D}_{IV} \subseteq \mathbb{C}^n$ with $n \geq 3$. The decomposition of $\mathcal{P}(\mathbb{C}^n)$ into irreducible $\mathrm{SO}(n-1) \times \mathrm{SO}(2)$-submodules is given by

$$\mathcal{P}(\mathbb{C}^n) = \bigoplus_{(k_1,k_2)\in\mathbb{N}^2} \bigoplus_{c_1=0}^{k_1} V_{k_1,k_2}^{c_1}$$

such that some $V_{k_1,k_2}^{c_1}$ appear more than once. In particular, the representation has higher multiplicity and thus the Toeplitz C^*-algebras $\mathscr{T}_\lambda(\mathcal{A}^{\mathrm{SO}(n-1)\times\mathrm{SO}(2)})$ are not commutative.

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Proof:

$$V_{1,2}^1 \cong V_{3,1}^1 \text{ as } \mathrm{SO}(n-1) \times \mathrm{SO}(2)\text{-submodules.}$$

Referencias I

[BTD13] Theodor Bröcker and Tammo Tom Dieck. *Representations of compact Lie groups*. Vol. 98. Springer Science & Business Media, 2013.

[DÓQB15] Matthew Dawson, Gestur Ólafsson, and Raúl Quiroga-Barranco. "Commuting Toeplitz operators on bounded symmetric domains and multiplicity-free restrictions of holomorphic discrete series". In: *Journal of Functional Analysis* 268.7 (2015), pp. 1711–1732.

[DQB18] Matthew Dawson and Raul Quiroga-Barranco. "Radial Toeplitz operators on the weighted Bergman spaces of Cartan domains". In: *Representation Theory and Harmonic Analysis on Symmetric Spaces* 714 (2018), pp. 97–114.

[Hel79] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces*. Academic press, 1979.

Referencias II

[Joh80] Kenneth D Johnson. "On a ring of invariant polynomials on a Hermitian symmetric space". In: *Journal of Algebra* 67.1 (1980), pp. 72–81.

[Kna01] Anthony W Knapp. *Representation theory of semisimple groups: an overview based on examples*. Vol. 36. Princeton university press, 2001.

[Kna02] Anthony W Knapp. "Lie groups beyond an introduction. Second. Vol. 140". In: *Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA* (2002).

[Kob08] T. Kobayashi. *Representation theory and automorphic forms. Multiplicity-free Theorems of the Restrictions of Unitary Highest Weight Modules with respect to Reductive Symmetric Pairs*. Vol. 255. Birkhäuser, 2008, pp. 45–109.

Referencias III

[Kor68] Adam Korányi. "Analytic invariants of bounded symmetric domains". In: *Proceedings of the American Mathematical Society* 19.2 (1968), pp. 279–284.

[Mok89] Ngaiming Mok. *Metric rigidity theorems on Hermitian locally symmetric manifolds*. Vol. 6. World Scientific, 1989.

[Nar71] Raghavan Narasimhan. *Several complex variables*. University of Chicago Press, 1971.

[Upm12] Harald Upmeier. *Toeplitz operators and index theory in several complex variables*. Vol. 81. Birkhäuser, 2012.

Thank You!